§7. Motivation for Maass forms

In the rest of the semester, we'll look at the case of Maass forms.

These are functions that transform like a modular form but we do not require anymore that they are holomorphic.

Getting rid of this strong regularity condition will also lead to non-trivial examples even in weight zero.

Recall: $M_0(\Gamma) = \mathbb{C}$

whereas when we allowed meromorphicity at ∞ we already got a very interesting non-trivial example $j(z) = E_4^3/\Delta$

$$= \frac{1}{q} + 744 + 196884\, q + \cdots$$

Maass forms are functions $f: \mathbb{H} \to \mathbb{C}$ that are ① invariant under Γ: $f(\gamma z) = f(z)$ $\forall \gamma \in \Gamma$

② $\Delta f = \lambda f$ for some $\lambda \in \mathbb{C}$

where $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ is the so-called hyperbolic Laplacian

③ For any x, $f(x+iy) = O(y^N)$ for some N, as $y \to \infty$

Ie. f satisfies a moderate growth condition at "∞"

But to motivate their study we'll first look at a very classical problem of representation of integers by quadratic forms and Dirichlet's class # formula

§7.1 Class number formula

Dirichlet's class number formula in its simplest form was conjectured by Jacobi in 1832 and proved by Dirichlet in 1839. Nowadays it is given in terms of quadratic fields and their class number. The formula of Dirichlet was generalized by Dedekind to arbitrary number fields and it relates arithmetic data associated to a finite extension $K/\mathbb{Q}$ to the residue at $s=1$ of the zeta function $\zeta_K(s)$ associated to K

Let $[K:\mathbb{Q}] = n = r_1 + 2r_2$

where r_1 = # of real embeddings of K

$2r_2$ = # of complex " of K.

Let $\zeta_K(s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s}$ where $\mathfrak{a}$ runs through non-zero ideals in the ring of integers of K $\mathcal{O}_K$ of K

$N_{K/\mathbb{Q}}(\mathfrak{a})$ is the norm of the ideal which is equal to $[\mathcal{O}_K : \mathfrak{a}]$

$\mathcal{O}_K$ = ring of alg. integers contained in K

$= \{ \alpha \in K \mid \alpha$ is a root of a monic poly in $\mathbb{Z}[x] \}$

Let h_K = class number = # of elts in the ideal class group of K.

(Class group $= J_K / P_K$

J_K = group of fractional ideals of $\mathcal{O}_K$

P_K = group of principal ideals

[A fractional ideal of an integral domain R (with K = its field of fractions)

is an R-submodule I of K such that

$\exists\, r \in R$ st $rI \subseteq R$

(A fractional ideal $I \subseteq R \Leftrightarrow$ it is an integral ideal of R)

A principal fractional ideal is an R-submodule of K generated by a single elt of K.

Let D_K = discriminant of $K/\mathbb{Q}$

Let Reg_K = regulator of K

= This is the determinant of a $r \times r$ minor of a $r \times (r+1)$ matrix where $r = r_1 + r_2 - 1$

The entries of the matrix M formed by taking $u_1, \dots u_r$, a set of generators of the unit gp of K and $r+1$ different embeddings of K into $\mathbb{R}$ or $\mathbb{C}$.

If $K = \mathbb{Q}(\sqrt{D})$ for $D < 0$

$R_K = 1$

$$M = \left(\alpha_j \log |u_i^{(j)}| \right) \quad \begin{array}{l} i = 1, \dots r \\ j = 1, \dots r+1 \end{array}$$

$\alpha_j = 1$ or 2 if j-th embed. is $\mathbb{R}$ or $\mathbb{C}$.

w_K = number of roots of unity in K

Then

<u>Thm</u> (Class # formula) $\zeta_K(s)$ converges abs for $\mathrm{Re}\, s > 1$ and extends to a meromorphic function defined $\forall s \in \mathbb{C}$ with only a simple pole at $s = 1$ with residue

$$\lim_{s \to 1} (s-1)\, \zeta_K(s) = \frac{2^{r_1} (2\pi)^{r_2}\, \mathrm{Reg}_K\, h_K}{w_K \cdot |D_K|}$$

We'll take a more elementary approach and prove Dirichlet's class # formula for integral primitive Binary quadratic forms of negative discriminant. There is a correspondence between primitive quadratic forms of disc D and ideals in a quadratic field of disc D.

The classical formula of Dirichlet relates the class # of positive def. BQFs of disc. D to the value at $s=1$ of $L(s, \chi_D)$ for a character mod D.

and then expresses the value $L(1, \chi_D)$ as a finite sum.

We'll start with the first stage, which expresses $L(1, \chi_D)$ in terms of class # h_D.

For this we start with a quick overview of Binary quadratic forms.

Binary quadratic forms.

(ref: Buell, Zagier)

We start with a classical theorem of Fermat

<u>Thm</u> (Fermat) $x^2+y^2=p$ p a prime has a solution $(x,y)\in\mathbb{Z}^2 \iff p\equiv 1 \bmod 4$.

<u>Pf.</u> see any book on elementary # theory

<u>Rmk.</u> Using Fermat's thm for a $p\equiv 1\ (4)$ we can say $\exists\, x,y\in\mathbb{Z}$ s.t. $x^2+y^2=p$

But then $p^2=(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$

Hence Fermat's thm says that if $p\equiv 1\ (4)$ then p is the hypotenuse of a right angled triangle with sides x^2-y^2, $2xy$, p and hence Fermat's thm is a thm in Pythagorean tradition.

Fermat also showed that

$x^2+2y^2=p$ has integral solns $\iff p\equiv 1$ or 3 mod 8

(eg. $3^2+2.2^2=17$)

$x^2+3y^2=p$ " " $\iff p\equiv 1$ mod 3

(eg $2^2+3.1^2=7$)

All these are examples of Binary quadratic forms (BQF)

$$Q(x,y) = Ax^2 + Bxy + Cy^2, \quad A, B, C \in \mathbb{Z}$$

(are relatively prime. if primitive)

Let $D = \operatorname{disc} Q = B^2 - 4AC$

We denote $Q_D := \left\{ Ax^2 + Bxy + Cy^2 \;\middle|\; \begin{array}{l} A, B, C \in \mathbb{Z} \\ B^2 - 4AC = D \end{array} \right\}$

and will write $[A, B, C]$ for $Ax^2 + Bxy + Cy^2$

The question whether a given prime p can be written as $Q(x,y) = p$ for some $Q = [A, B, C]$ and $x, y \in \mathbb{Z}$ has some redundancy.

For example the question whether $2x^2 + 3y^2 = p$ has integral soln is same as $2y^2 + 3x^2 = p$

This is just change of variables $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Less obvious is that it is also same if we consider the quadratic form

$$2x^2 + 4xy + 5y^2$$

Since $2x^2 + 4xy + 5y^2 = 2(x+y)^2 + 3y^2$

and this is the change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix}$$

To avoid this kind of redundancy Gauss introduced an equivalence relation for quadratic forms.

Defn We say 2 forms $Q=[A,B,C]$ and $Q'=[A',B',C']$ are equivalent and write $Q\sim Q'$ if $\exists$ a matrix $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,\mathbb{Z})$ s.t

$$Q'(x,y)=Q(\alpha x+\beta y,\ \gamma x+\delta y)$$

Ex show that this defines an equivalence relation on the set of quadratic forms.

Defn We say a quadratic form $Q=[a,b,c]$ represents an integer n, if $\exists\ (x,y)\in\mathbb{Z}^2$ s.t. $Q(x,y)=n$, ie n is in the range of Q seen as a function $Q:\mathbb{Z}^2\to\mathbb{Z}$.

Note simple arithmetic shows that

$$(*)\quad 4A(Ax^2+Bxy+Cy^2)=(2Ax+By)^2+(4AC-B^2)y^2$$

Hence if $D=B^2-4AC<0$, then the RHS of $(*)$ is always positive. Hence $\mathrm{sgn}(A)=\mathrm{sgn}(Ax^2+Bxy+Cy^2)$.

Hence a form Q with negative disc $D<0$ represents only positive numbers or only negative numbers.

Clearly the range of values of $[A, B, C]$ is the negative of range of values of $[-A, -B, -C]$.

From now on for $D<0$ we consider only the forms that represent positive numbers. Such forms are called <u>positive definite</u>. They have $A>0$.

Since $[A, B, C] \sim [C, -B, A]$ via $(x, y) \to (-y, x)$

if $[A, B, C]$ is positive definite, so is $[C, -B, A]$. Hence $C>0$ as well.

<u>Rmk.</u> Every quadratic form $Q(x,y) = Ax^2 + Bxy + Cy^2$ can be written in matrix form

$$Q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If $Q' \sim Q$ via a matrix $M \in SL_2(\mathbb{Z})$

i.e. $Q'(x,y) = Q((x\ y)M^t)$

then $Q'(x,y) = \begin{pmatrix} x & y \end{pmatrix} M^t \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} M \begin{pmatrix} x \\ y \end{pmatrix}$

Hence $\begin{pmatrix} A' & B'/2 \\ B'/2 & C' \end{pmatrix} = M^t \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} M$

Note that if $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = M \in SL(2,\mathbb{Z})$

and $\begin{pmatrix}\alpha x + \beta y \\ \gamma x + \delta y\end{pmatrix} = \begin{pmatrix}x' \\ y'\end{pmatrix} = M\begin{pmatrix}x \\ y\end{pmatrix}$

then a quadratic form $Q = [a, b, c]$ is equivalent to $Q' = [a', b', c'] = Q((x\ y)M^t)$

where
$$a'x^2 + b'xy + c'y^2 = a(\alpha x + \beta y)^2 + b(\alpha x + \beta y)(\gamma x + \delta y) + c(\gamma x + \delta y)^2$$

Hence
$$a' = a\alpha^2 + b\alpha\gamma + c\gamma^2 = Q(\alpha, \gamma)$$
$$b' = 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta$$
$$c' = a\beta^2 + b\beta\delta + c\delta^2 = Q(\beta, \delta)$$

Question How many equivalence classes of quadratic forms are there?

Answer Clearly infinite since the discriminant $b^2 - 4ac$ remain invariant under the action of $M \in SL(2,\mathbb{Z})$ (Check!)

ie if $Q \sim Q'$ with $M \in SL(2,\mathbb{Z})$ then $b^2 - 4ac = b'^2 - 4a'c'$.

And there are ∞ly many integers $D \equiv 0, 1\ (4)$

Note $D \equiv 0, 1 \mod 4$ since

$$D = b^2 - 4ac \equiv b^2 \mod 4$$ and

mod 4 any square is 0 or 1 mod 4.

Conversely for any $D \equiv 0, 1\ (4)$ there is

at least one form of disc D namely

$$Q_0(x, y) := \begin{cases} x^2 - D/4\, y^2 & \text{if } D \equiv 0\ (4) \\ x^2 + xy + \frac{1-D}{4} & \text{if } D \equiv 1\ (4) \end{cases}$$

$Q_1(x, y)$ is called the principal form

A better question is if we fix the discriminant

D, and look at $\mathcal{Q}_D = \left\{ [a, b, c] \,\middle|\, \begin{array}{l} a, b, c \in \mathbb{Z} \\ b^2 - 4ac = D \end{array} \right\}$

Then $SL(2, \mathbb{Z})$ also acts on $\mathcal{Q}_D$

Question How many equivalence classes are there? Is it finite?

This is answered by Lagrange who showed that

Thm 7.1 Let $D \in \mathbb{Z}$, $D \neq \square$. Then

$\exists$ finitely many equivalence classes in

$\Gamma \backslash \mathcal{Q}_D$.

The proof of Thm 7.1 is based on the following so called reduction thm.

(Lagrange)
Thm 7.2 Let $Q = [a, b, c] \in \mathcal{Q}_D$ then $\exists\, Q' = [a', b', c'] \in \mathcal{Q}_D$ s.t. $Q \sim Q'$ and coefs of Q' satisfy

$$|b'| \leq |a'| \leq |c'| \quad (*)$$

Proof of Thm 7.1 follows from $(*)$ in Thm 7.2 since there are only finitely many $[a', b', c'] \in \mathcal{Q}_D$ that satisfy $(*)$. Indeed if $[a', b', c']$ so satisfy $(*)$ and $b'^2 - 4a'c' = D$

Then $|D| = |b'^2 - 4a'c'| \geq |4a'c'| - |b'|^2$
$$\underset{(*)}{\geq} 4|a'|^2 - |a'|^2 = 3a'^2$$

Hence $\boxed{|a'| \leq \sqrt{\frac{|D|}{3}}}$

This gives finitely many choices for a'. Since $|b'| \leq |a'|$ each a' gives rise to finitely many b' and since $c' = \frac{b'^2 - D}{4a'}$ also to finitely many c'. ☒